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O(d,d) invariance at two and three loops

I. Jack and S. Parsons

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, United Kingdom

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We show that in a two-dimensional σ model whose fields only depend on one target space coordinate, the $O(d,d)$ invariance of the conformal invariance conditions observed at one loop is preserved at two loops (in the general case with torsion) and at three loops (in the case without torsion).

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I. INTRODUCTION

Duality invariance has proved to be an immensely fruitful concept in string theory. A particular aspect of duality which has been valuable in string cosmology (for a recent comprehensive review of this subject, with an extensive list of references, see Ref. [1]) is the concept of $O(d,d)$ invariance. This is displayed in the case where the fields depend only on time in an arbitrary number of dimensions (time and d spatial dimensions). It was discovered some time ago [2] that the lowest order string effective action (in the simplest case with only the metric, dilaton and antisymmetric tensor field) then exhibits continuous, global $O(d,d)$ invariance. [This is also reminiscent of the $O(d,d)$ invariance previously observed in the context of string compactification [3].] The $O(d,d)$ invariance of the string action was later found to persist in the presence of matter or gauge fields [4]. Now the conformal invariance conditions for the fields are related to the field equations of the string action, and so, at least at lowest order, the $O(d,d)$ invariance can be used to transform between different conformal backgrounds. Duality invariance in general can be understood as a consequence of an isometry in an underlying theory [5]; and the $O(d,d)$ invariance can be viewed as the result of gauging d abelian isometries [6]. In Refs. [2] and [7] it was argued that the $O(d,d)$ invariance should be maintained to all orders. In Ref. [8] the two-loop string action was considered in the context of fields depending only on time, and it was shown that after a suitable field redefinition it could be written in an explicitly $O(d,d)$ invariant form. However, it was pointed out in Refs. [9] (in the context of T duality) that the invariance of the action does not manifestly guarantee that the conformal invariance conditions transform appropriately. Therefore it seems to us that it is necessary to check explicitly the transformation properties of the conformal invariance conditions in order to complete the proof that $O(d,d)$ invariance is preserved. At one loop this has been done in Ref. [2]. The main purpose of this paper is to verify the invariance at two loops (with torsion) and at three loops (without torsion). We shall find various subtleties which do not arise in checking the invariance of the action.

II. THE ONE-LOOP CASE

The two-dimensional nonlinear σ model is defined by the action

$$S=\frac{1}{2\pi\alpha'}\int d^2x\{\sqrt{\gamma}g_{\mu\nu}\partial_\alpha x^\mu\partial^\alpha x^\nu+\epsilon^{\alpha\beta}b_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu -2\alpha'\sqrt{\gamma}R^{(2)}\phi\}, \tag{2.1}$$

where α' is the usual string coupling, which we shall henceforth set to 1 for convenience, $g_{\mu\nu}$ is the metric, $b_{\mu\nu}$ is the antisymmetric tensor field, and ϕ is the dilaton. $\gamma_{\alpha\beta}$ is the metric on the two-dimensional worldsheet, $\gamma=\det\gamma_{\alpha\beta}$, $\epsilon^{\alpha\beta}$ is the two-dimensional alternating symbol and $R^{(2)}$ is the worldsheet Ricci scalar. Conformal invariance for the σ model requires the vanishing of the three functions $\bar{\beta}^g$, $\bar{\beta}^b$, and $\bar{\beta}^\phi$, which are defined by [10]

$$\begin{aligned}\bar{\beta}^g_{\mu\nu}&=\beta^g_{\mu\nu}+2\nabla_\mu\partial_\nu\phi+2\nabla_{(\mu}S_{\nu)},\\ \bar{\beta}^b_{\mu\nu}&=\beta^b_{\mu\nu}+H^\rho{}_{\mu\nu}\partial_\rho\phi+H^\rho{}_{\mu\nu}S_\rho,\\ \bar{\beta}^\phi&=\beta^\phi+(\partial\phi)^2+\partial^\rho\phi S_\rho.\end{aligned} \tag{2.2}$$

Here β^g, β^b, and β^ϕ are the renormalization group β functions for the σ model, and $H_{\mu\nu\rho}$ is the field strength for $b_{\mu\nu}$, defined by $H_{\mu\nu\rho}=3\nabla_{[\mu}b_{\nu\rho]}$. The vector S^μ arises in the process of defining the trace of the energy-momentum tensor as a finite composite operator, and can be computed perturbatively. At one loop we have

$$\begin{aligned}\beta^{g(1)}_{\mu\nu}&=R_{\mu\nu}-\frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma},\\ \beta^{b(1)}_{\mu\nu}&=-\frac{1}{2}\nabla^\rho H_{\rho\mu\nu},\\ \beta^{\phi(1)}&=-\frac{1}{2}\nabla^2\phi-\frac{1}{24}H^2,\end{aligned} \tag{2.3}$$

where $H^2=H_{\kappa\lambda\rho}H^{\kappa\lambda\rho}$, and $S^\mu=0$. The conformal invariance conditions can be derived at this order from the string effective action

$$\Gamma^{(1)}=\int d^{d+1}x\sqrt{-g}e^{-2\phi}\{R-\tfrac{1}{12}H^2+4(\partial\phi)^2\}. \tag{2.4}$$

To be more precise, we have

$$\frac{\partial\Gamma^{(1)}}{\partial g^{\mu\nu}}=\overline{\beta}^{g(1)}_{\mu\nu}+g_{\mu\nu}\tilde{\beta}^{\phi},$$

$$\frac{\partial\Gamma^{(1)}}{\partial\beta^{\mu\nu}}=-\overline{\beta}^{b(1)}_{\mu\nu}, \tag{2.5}$$

$$\frac{\partial\Gamma^{(1)}}{\partial\phi}=4\tilde{\beta}^{\phi(1)},$$

where

$$\tilde{\beta}^{\phi}=2\overline{\beta}^{\phi}-\tfrac{1}{2}\,g^{\mu\nu}\overline{\beta}^{g}_{\mu\nu}. \tag{2.6}$$

To consider $O(d,d)$ duality, we specialize to a metric with signature $(-,+,+,...,+)$ and we consider a σ model where the fields depend only on the first coordinate t. We can then bring g and b to the block-diagonal form

$$g_{\mu\nu}=\begin{pmatrix} g_{00} & 0\\ 0 & G(t)\end{pmatrix},\quad b_{\mu\nu}=\begin{pmatrix} 0 & 0\\ 0 & B(t)\end{pmatrix}. \tag{2.7}$$

[For the discussion of $O(d,d)$ invariance at one loop, one can take $g_{00}=-1$; but at higher loops we need to consider a general g_{00} at intermediate stages, returning to $g_{00}=-1$ at the end of the computation. In fact, we shall only retain a general g_{00} at points where it will leave an imprint even after setting $g_{00}=-1$—for instance, where it is acted upon by a $\partial/\partial g_{00}$.] It was shown in Ref. [2] that the one-loop action may then be written as

$$\Gamma^{(1)}=-\int dt e^{-\Phi}g_{00}^{-1/2}(\dot{\Phi}^2+\tfrac{1}{8}\,\mathrm{Tr}[\dot{M}\eta\dot{M}\eta]), \tag{2.8}$$

where

$$\Phi=2\phi-\tfrac{1}{2}\ln\det G, \tag{2.9}$$

η is the metric for the $O(d,d)$ group in nondiagonal form given by

$$\eta=\begin{pmatrix} 0_d & 1_d\\ 1_d & 0_d\end{pmatrix} \tag{2.10}$$

and

$$M=\begin{pmatrix} G^{-1} & -G^{-1}B\\ BG^{-1} & G-BG^{-1}B\end{pmatrix}. \tag{2.11}$$

The $O(d,d)$ group is represented by matrices Ω such that

$$\Omega\,\eta\Omega^T=\eta; \tag{2.12}$$

the action in Eq. (2.8) is invariant under the action of the $O(d,d)$ group with

$$M\to M'=\Omega^T M\Omega,\quad \Phi\to\Phi,\quad g_{00}\to g_{00}. \tag{2.13}$$

The matrix M has two important properties: M is symmetric and $M\in O(d,d)$.

As we mentioned earlier, the invariance of the action does not manifestly guarantee that the conformal invariance conditions transform appropriately. In the $O(d,d)$ case, even at one loop, where the conformal invariance conditions are simply related to the action by Eq. (2.5), it does not seem immediately obvious how the correct properties for the $\overline{\beta}$ follow from the invariance of the action; and at higher loops, where the relation of the $\overline{\beta}$ functions to the action is more complicated, it is still less clear. Therefore in this paper we shall explicitly check the transformation properties of the $\overline{\beta}$ functions.

As in the case of the action, the β functions are most conveniently discussed in terms of the matrix M. Defining $\beta^M=\mu(d/d\mu)M$, we have, from Eq. (2.11),

$$\beta^M=\begin{pmatrix} -G^{-1}\beta^G G^{-1} & G^{-1}(\beta^G G^{-1}B-\beta^B)\\ (\beta^B-BG^{-1}\beta^G)G^{-1} & \beta^G-\beta^B G^{-1}B+BG^{-1}\beta^G G^{-1}B-BG^{-1}\beta^B\end{pmatrix}. \tag{2.14}$$

We then define

$$\overline{\beta}^M=\beta^M-\tfrac{1}{2}\,(\dot{\phi}+S)\dot{M}=\begin{pmatrix} -G^{-1}\overline{\beta}^G G^{-1} & G^{-1}(\overline{\beta}^G G^{-1}B-\overline{\beta}^B)\\ (\overline{\beta}^B-BG^{-1}\overline{\beta}^G)G^{-1} & \overline{\beta}^G-\overline{\beta}^B G^{-1}B+BG^{-1}\overline{\beta}^G G^{-1}B-BG^{-1}\overline{\beta}^B\end{pmatrix}, \tag{2.15}$$

where we assume (as will always be the case in the present calculation) that $S_i=0$ and $S_0\equiv S$.

We start with the one-loop $\overline{\beta}$ functions. As mentioned before, these have already been demonstrated to have the correct properties, but we shall repeat the check in order to demonstrate our formalism in operation. Upon specializing the general results in Eqs. (2.2), (2.3) to the forms for $g_{\mu\nu}$ and $b_{\mu\nu}$ given in Eq. (2.7), we find that the $\overline{\beta}$ functions for G and B are given at one loop by

$$\overline{\beta}^{G(1)}=\tfrac{1}{2}\,(-g^{00})G(X-W^2-\tilde{W}^2-\dot{\Phi}W-\tfrac{1}{2}\,g^{00}\dot{g}_{00}W), \tag{2.16}$$
$$\overline{\beta}^{B(1)}=\tfrac{1}{2}\,(-g^{00})G(\tilde{X}-W\tilde{W}-\tilde{W}W-\dot{\Phi}\tilde{W}-\tfrac{1}{2}\,g^{00}\dot{g}_{00}\tilde{W}),$$

where

$$W=G^{-1}\dot{G},\quad \tilde{W}=G^{-1}\dot{B},\quad X=G^{-1}\ddot{G},\quad \tilde{X}=G^{-1}\ddot{B}. \tag{2.17}$$

As mentioned earlier, to demonstrate the invariance at two and three loops, we need to retain a general g_{00} for the moment, though we shall set $g_{00}=-1$ at the end of the calculation. Substituting Eq. (2.16) in Eq. (2.15), we find that we can write

$$\overline{\beta}^{M(1)}=\tfrac{1}{2}(\ddot{M}+\dot{M}\eta\dot{M}\eta M-\dot{\Phi}\dot{M}). \tag{2.18}$$

(Expressions for $\dot{M}$ and $\ddot{M}$ are given in the Appendix.) The $O(d,d)$ invariance for $\overline{\beta}^G_{ij}$ and $\overline{\beta}^B_{ij}$ is then manifest, for clearly when M and Φ transform according to Eq. (2.13), $\overline{\beta}^{M(1)}$ transforms according to

$$\overline{\beta}^M\to\overline{\beta}^M(M')=\Omega^T\overline{\beta}^M(M)\Omega. \tag{2.19}$$

Now if Eq. (2.19) is satisfied, then since from Eqs. (2.13), (2.15), we also have $\overline{\beta}^{M'}(M')=\Omega^T\overline{\beta}^M(M)\Omega$, we deduce that $\overline{\beta}^M(M)=\overline{\beta}^{M'}(M)$, i.e., $\overline{\beta}^M$ is form invariant under the $O(d,d)$ transformation. A solution of the conformal invariance conditions with $\overline{\beta}^M(M)=0$ automatically satisfies $\overline{\beta}^{M'}(M')=0$; but now we also have $\overline{\beta}^M(M')=0$, i.e., the transformed M' is a solution of the *same* conformal invariance conditions as M. Equation (2.19) will be our touchstone for $O(d,d)$ invariance at two and three loops as well.

We also find

$$\overline{\beta}^{g(1)}_{00}=\ddot{\Phi}+\tfrac{1}{8}\mathrm{Tr}[\dot{M}\eta\dot{M}\eta]-\tfrac{1}{2}g^{00}\dot{g}_{00}\dot{\Phi} \tag{2.20}$$

and

$$\begin{aligned}\overline{\beta}^{\Phi(1)}&=2\overline{\beta}^{\phi(1)}-\tfrac{1}{2}G^{ij}\overline{\beta}^{G(1)}_{ij}\\&=(-g^{00})(\tfrac{1}{2}\ddot{\Phi}-\tfrac{1}{2}\dot{\Phi}^2-\tfrac{1}{4}g^{00}\dot{g}_{00}\dot{\Phi}).\end{aligned} \tag{2.21}$$

We note here that the trace of an even number of products of $M\eta$ and its derivatives is manifestly invariant under Eq. (2.13) while the trace of an odd number of such products is zero, and therefore $\overline{\beta}^{g(1)}_{00}$ and $\overline{\beta}^{\Phi(1)}$ are $O(d,d)$ invariant.

III. THE TWO-LOOP CASE

In this section we shall show that Eq. (2.19) continues to hold, and that $\overline{\beta}^{g(1)}_{00}$ and $\overline{\beta}^{\Phi(1)}$ are $O(d,d)$ invariant, at two loops. At this order, however, we find that to make the invariance manifest we need to make field redefinitions, as has already been found in the case of the two-loop action in Ref. [8].

The two-loop β functions are given by [11]

$$\begin{aligned}\beta^{g(2)}_{\mu\nu}&=\tfrac{1}{2}R_{\mu\kappa\rho\sigma}R_\nu{}^{\kappa\rho\sigma}+\tfrac{1}{24}\nabla_\mu H_{\kappa\rho\sigma}\nabla_\nu H^{\kappa\rho\sigma}+\tfrac{1}{8}(1-2\lambda)\nabla_\kappa H_{\mu\rho\sigma}\nabla^\kappa H_\nu{}^{\rho\sigma}+\tfrac{1}{4}\lambda R^\kappa{}_{\mu\nu\lambda}H_{\kappa\rho\sigma}H^{\lambda\rho\sigma}+\tfrac{1}{2}(3-2\lambda)R^\kappa{}_{\lambda\rho\mu}H_{\nu\sigma\kappa}H^{\lambda\rho\sigma}\\&\quad+\tfrac{1}{2}(\lambda-1)R_{\kappa\lambda\rho\sigma}H_\mu{}^{\kappa\rho}H_\nu{}^{\lambda\sigma}+\tfrac{1}{16}H_{\kappa\lambda\rho}H_\sigma{}^{\lambda\rho}H_{\mu\tau}{}^\kappa H_\nu{}^{\tau\sigma}+\tfrac{1}{16}H_{\mu\kappa\lambda}H_{\nu\rho\sigma}H_\tau{}^{\kappa\rho}H^{\tau\lambda\nu},\\
\beta^{b(2)}_{\mu\nu}&=-R_{\kappa\lambda\rho\mu}\nabla^\lambda H_\nu{}^{\kappa\rho}+\tfrac{1}{8}\nabla_\kappa(H_{\mu\lambda\rho}H_\nu{}^{\lambda\sigma})H_\sigma{}^{\kappa\rho}-\tfrac{1}{4}(\lambda-1)\nabla_\kappa(H_\sigma{}^{\lambda\rho}H_{\lambda\rho\mu})H_\nu{}^{\sigma\kappa}+\tfrac{1}{8}\lambda\nabla_\kappa H_{\mu\nu\sigma}H_{\lambda\rho}{}^\kappa H^{\sigma\lambda\rho},\\
\beta^{\phi(2)}&=-\tfrac{1}{8}R_{\kappa\lambda\rho\sigma}R^{\kappa\lambda\rho\sigma}-\tfrac{1}{24}(1+6\mu)\nabla_\kappa H_{\lambda\rho\sigma}\nabla^\kappa H^{\lambda\rho\sigma}+\tfrac{1}{8}(5+12\mu)R_{\kappa\lambda\rho\sigma}H_\tau{}^{\lambda\sigma}H^{\tau\kappa\rho}-\tfrac{1}{16}(\tfrac{1}{4}+\lambda+3\mu)H_{\kappa\rho\sigma}H_\lambda{}^{\rho\sigma}H^{\kappa\tau\omega}H_{\tau\omega}{}^\lambda\\&\quad-\tfrac{5}{192}H_{\kappa\lambda\rho}H_{\sigma\tau}{}^\kappa H_\omega{}^{\lambda\sigma}H^{\omega\rho\tau}+\tfrac{1}{8}\lambda\nabla_\kappa\nabla_\lambda\phi H^{\kappa\rho\sigma}H_{\rho\sigma}{}^\lambda,\end{aligned} \tag{3.1}$$

with

$$S_\mu=-\tfrac{1}{24}(\lambda+3\mu)\nabla_\mu(H_{\rho\sigma\tau}H^{\rho\sigma\tau}). \tag{3.2}$$

Here, λ and μ represent the effects of field redefinitions of the form

$$\delta g_{\mu\nu}=\lambda H_{\mu\rho\sigma}H^{\rho\sigma}_\nu,\qquad \delta\phi=-\tfrac{1}{2}\mu H_{\rho\sigma\tau}H^{\rho\sigma\tau}. \tag{3.3}$$

With $g_{\mu\nu}$ and $b_{\mu\nu}$ as given by Eq. (2.7), we find, using identities given in the Appendix,

$$\begin{aligned}\overline{\beta}^{G(2)}&=\tfrac{1}{32}\{-4XW^2+4X^2+4\lambda W\tilde{W}^2W+8(1-\lambda)W^2\tilde{W}^2-8(1-\lambda)W\tilde{W}W\tilde{W}+4(1-2\lambda)\tilde{X}(W\tilde{W}+\tilde{W}W)-8(1-2\lambda)\tilde{X}^2\\&\quad-4(3-2\lambda)X\tilde{W}^2-8(1-\lambda)\tilde{W}X\tilde{W}+4(1-\lambda)\tilde{W}W^2\tilde{W}+4\tilde{W}^4-(1-2\lambda)(\mathrm{tr}[\tilde{W}^2]W^2+\mathrm{tr}[W^2]\tilde{W}^2)+\mathrm{tr}[W^2]W^2\\&\quad+\mathrm{tr}[\tilde{W}^2]\tilde{W}^2-2\lambda\,\mathrm{tr}[\tilde{W}^2]X-2\lambda\,\mathrm{tr}[\tilde{W}^2W]W+2(\lambda+3\mu)\partial_0\mathrm{tr}(\tilde{W}^2)W\}+\text{transpose},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\overline{\beta}^{B(2)}&=\tfrac{1}{8}\{2X\tilde{X}-W^2\tilde{X}-X(W\tilde{W}+\tilde{W}W)+2\lambda W\tilde{W}^3+(1-2\lambda)\tilde{X}\tilde{W}^2+2(1-\lambda)\tilde{W}\tilde{X}\tilde{W}-2(1-\lambda)\tilde{W}W\tilde{W}^2+\tfrac{1}{2}\mathrm{tr}[W^2]W\tilde{W}\\&\quad-\tfrac{1}{2}(1-2\lambda)\mathrm{tr}[\tilde{W}^2]W\tilde{W}-\tfrac{1}{2}\lambda\,\mathrm{tr}[\tilde{W}^2]\tilde{X}-\tfrac{1}{2}\lambda\,\mathrm{tr}[W\tilde{W}^2]\tilde{W}+\tfrac{1}{2}(\lambda+3\mu)\partial_0\mathrm{tr}(\tilde{W}^2)\tilde{W}\}-\text{transpose},\end{aligned} \tag{3.5}$$

$$\overline{\beta}_{00}^{g(2)} = \tfrac{1}{4}\mathrm{Tr}[-X^2+XW^2-\tfrac{1}{4}W^4+(1-\lambda)\tilde{X}^2-(1-\lambda)(\tilde{X}W\tilde{W}+\tilde{X}\tilde{W}W)+(1-\lambda)W\tilde{W}W\tilde{W}-W^2\tilde{W}^2$$
$$+(3-\lambda)X\tilde{W}^2-\tfrac{3}{4}\tilde{W}^4-(\lambda+3\mu)\partial_0^2(\tilde{W}^2)], \tag{3.6}$$

$$\overline{\beta}^{\Phi(2)} = \tfrac{1}{128}\mathrm{Tr}[\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta]-\tfrac{1}{256}(\mathrm{Tr}[\dot{M}\eta\dot{M}\eta])^2-\tfrac{1}{4}(\lambda+3\mu)\mathrm{Tr}[\tilde{X}^2+2X\tilde{W}^2+W\tilde{W}W\tilde{W}-\tilde{X}(W\tilde{W}+\tilde{W}W)$$
$$-W^2\tilde{W}^2-\tilde{W}^4+\tfrac{1}{4}\mathrm{tr}[\tilde{W}^2](\mathrm{tr}[W^2]-\mathrm{tr}[\tilde{W}^2])-\mathrm{tr}[\tilde{X}\tilde{W}-\tilde{W}^2W]\dot{\Phi}]-\tfrac{1}{8}\lambda(\mathrm{tr}[\tilde{W}^2]\ddot{\Phi}+\mathrm{tr}[\tilde{W}^2W]\dot{\Phi}). \tag{3.7}$$

The $O(d,d)$ invariance is not immediately manifest at two loops; for instance, $\overline{\beta}_{00}^{g(2)}$ and $\overline{\beta}^{\Phi(2)}$ cannot be written in terms of the traces of products of an even number of $M\eta$ and its derivatives. However, we may take advantage of the possibility of redefining the fields by

$$(G_r)_{ij}=G_{ij}+\delta G_{ij},$$
$$(B_r)_{ij}=B_{ij}+\delta B_{ij},\quad (g_r)_{00}=g_{00}+\delta g_{00}, \tag{3.8}$$
$$\Phi_r=\Phi+\delta\Phi,\qquad S_r=S+\delta S,$$

with M_r, β^{M_r}, and $\overline{\beta}^{M_r}$ defined as in Eqs. (2.11), (2.14), and (2.15), but with G replaced by G_r, etc. Of course, in order to maintain the global $O(d,d)$ symmetry, the variations in Eq. (3.8) should only depend on t. Note that here again we have included a general g_{00}. The idea that duality invariance might require corrections at higher orders was put forward in Ref. [12], and an early example in the current context for a particular string background was given in Ref. [13]. The two-loop corrections required for T duality of the general string effective action were obtained in Ref. [14], and the invariance of the β functions was discussed in the torsion-free case in Refs. [9], [15].

The changes in Eq. (3.8) induce corresponding modifications in the $\overline{\beta}$ functions according to

$$\delta\overline{\beta}^G=\left(\beta^G_{kl}\frac{\partial}{\partial G_{kl}}+\beta^g_{00}\frac{\partial}{\partial g_{00}}+\beta^\beta_{kl}\frac{\partial}{\partial B_{kl}}\right)\delta G-\Delta\beta^G+(\delta S+\tfrac{1}{2}\delta\dot{\Phi})W,$$
$$\delta\overline{\beta}^B=\left(\beta^G_{kl}\frac{\partial}{\partial G_{kl}}+\beta^g_{00}\frac{\partial}{\partial g_{00}}+\beta^B_{kl}\frac{\partial}{\partial B_{kl}}\right)\delta B-\Delta\beta^B+(\delta S+\tfrac{1}{2}\delta\dot{\Phi})\tilde{W},$$
$$\delta\overline{\beta}^g_{00}=\left(\beta^G_{kl}\frac{\partial}{\partial G_{kl}}+\beta^B_{kl}\frac{\partial}{\partial B_{kl}}\right)\delta g_{00}-\Delta\beta^g_{00}-2(\delta\dot{S}+\tfrac{1}{2}\delta\dot{\Phi}), \tag{3.9}$$
$$\delta\overline{\beta}^\Phi=\left(\beta^G_{kl}\frac{\partial}{\partial G_{kl}}+\beta^B_{kl}\frac{\partial}{\partial B_{kl}}+\beta^g_{00}\frac{\partial}{\partial g_{00}}\right)\delta\Phi-\tfrac{1}{2}\delta\ddot{\Phi}-\tfrac{1}{4}\mathrm{tr}[W]\delta\dot{\Phi}-\tfrac{1}{2}\delta g^{00}[\ddot{\Phi}+\tfrac{1}{2}\mathrm{tr}[W]\dot{\Phi}]-\tfrac{1}{4}\delta\dot{g}_{00}\dot{\Phi}+\dot{\Phi}(\delta S+\tfrac{1}{2}\delta\dot{\Phi}),$$

where

$$\Delta\beta^G=\beta^G[G_r,B_r,(g_r)_{00}]-\beta^G(G,B,g_{00}), \tag{3.10}$$

and similarly for $\Delta\beta^B$, $\Delta\beta^g_{00}$. Note that we could restore $g_{00}=-1$ by making a coordinate redefinition; such a diffeomorphism leaves the $\overline{\beta}$ functions unaltered [10,16], and the overall changes in the $\overline{\beta}$ functions would therefore remain as given by Eq. (3.9). We should also mention at this point that we could alternatively (and equivalently) keep the fields G_{ij}, B_{ij}, Φ, and S fixed, and instead change the duality transformations Eq. (2.13). However, the approach we have adopted is more convenient calculationally.

Taking in Eq. (3.9)

$$\delta G=\tfrac{1}{4}(-g^{00})G[W^2-(1-2\lambda)\tilde{W}^2],\qquad \delta g_{00}=-\tfrac{1}{4}\lambda\,\mathrm{tr}[\tilde{W}^2],$$
$$\delta B=\tfrac{1}{4}(-g^{00})G(W\tilde{W}+\tilde{W}W),\qquad \delta\Phi=-\tfrac{1}{4}(\lambda+3\mu)(-g^{00})\mathrm{tr}[\tilde{W}^2], \tag{3.11}$$
$$\delta S=\tfrac{1}{16}\lambda(-g^{00})(2\,\mathrm{tr}[\tilde{X}\tilde{W}-2\tilde{W}^2W]+\mathrm{tr}[\tilde{W}^2]\mathrm{tr}[W]),$$

we find

$$\delta\overline{\beta}^{G(2)}=\tfrac{1}{8}\{-X^2+(1-2\lambda)\tilde{X}^2+2XW^2+2\lambda\tilde{X}W\tilde{W}+(1-\lambda)[2\tilde{W}X\tilde{W}+2X\tilde{W}^2-2\tilde{X}\tilde{W}W-2W^2\tilde{W}^2+2W\tilde{W}W\tilde{W}+W\tilde{W}^2W]$$
$$+(\lambda-2)\tilde{W}W^2\tilde{W}-W^4\}+\tfrac{1}{32}\{(1-2\lambda)(\mathrm{tr}[\tilde{W}^2]W^2+\mathrm{tr}[W^2]\tilde{W}^2)-\mathrm{tr}[W^2]W^2-\mathrm{tr}[\tilde{W}^2]\tilde{W}^2+2\lambda\,\mathrm{tr}[\tilde{W}^2]X$$
$$+2\lambda\,\mathrm{tr}[\tilde{W}^2W]W\}+\text{transpose},\tag{3.12}$$

$$\delta\overline{\beta}^{B(2)}=\tfrac{1}{4}\{-X\tilde{X}+\tilde{X}W^2+XW\tilde{W}+(1-\lambda)[-\tilde{X}\tilde{W}^2+\tilde{W}^3W+\tilde{W}^2W\tilde{W}-\tilde{W}\tilde{X}\tilde{W}]-W^3\tilde{W}\}+\tfrac{1}{8}\{-\tfrac{1}{2}\mathrm{tr}[W^2]W\tilde{W}$$
$$+\tfrac{1}{2}(1-2\lambda)\mathrm{tr}[\tilde{W}^2]W\tilde{W}+\tfrac{1}{2}\lambda\,\mathrm{tr}[\tilde{W}^2]\tilde{X}+\tfrac{1}{2}\lambda\,\mathrm{tr}[W\tilde{W}^2]\tilde{W}\}-\text{transpose},\tag{3.13}$$

$$\delta\overline{\beta}^{g(2)}_{00}=\tfrac{1}{8}\mathrm{tr}[2XW^2-2W^4-2(1-\lambda)X\tilde{W}^2-2(1+\lambda)\tilde{X}(\tilde{W}W+W\tilde{W})+4W^2\tilde{W}^2+(1+2\lambda)W\tilde{W}W\tilde{W}-\tilde{W}^4+2\lambda\tilde{X}^2],\tag{3.14}$$

$$\delta\overline{\beta}^{\Phi(2)}=-\tfrac{1}{4}(\lambda+3\mu)(\mathrm{tr}[-2X\tilde{W}^2+W^2\tilde{W}^2+\tilde{W}^4+\tilde{X}(W\tilde{W}+\tilde{W}W)-W\tilde{W}W\tilde{W}-\tilde{X}^2]+\dot{\Phi}\,\mathrm{tr}[\tilde{X}\tilde{W}-\tilde{W}^2W]$$
$$+\tfrac{1}{4}\mathrm{tr}[\tilde{W}^2-W^2]\mathrm{tr}[\tilde{W}^2])+\tfrac{1}{8}\lambda(\mathrm{tr}[\tilde{W}^2]\ddot{\Phi}+\mathrm{tr}[\tilde{W}^2W]\dot{\Phi}).\tag{3.15}$$

Combining Eqs. (3.4)–(3.7) with Eqs. (3.12)–(3.15), substituting in Eq. (2.15), and using the results in the Appendix, we now find that we have, up to this order,

$$\overline{\beta}^{M_r}=\tfrac{1}{2}(\ddot{M}_r+\dot{M}_r\eta\dot{M}_r\eta M_r-\dot{\Phi}\dot{M}_r)-\tfrac{1}{8}(\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r+\dot{M}_r\eta\dot{M}_r\eta\ddot{M}_r)-\tfrac{1}{4}\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta M_r,$$

$$\overline{\beta}^{g_r}_{00}=(\ddot{\Phi}+\tfrac{1}{8}\mathrm{Tr}[\dot{M}_r\eta\dot{M}_r\eta]-\tfrac{1}{2}g^{00}\dot{g}_{00}\dot{\Phi})+\tfrac{1}{8}\mathrm{Tr}[\ddot{M}_r\eta\ddot{M}_r\eta]-\tfrac{5}{32}\mathrm{Tr}[\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta],\tag{3.16}$$

$$\overline{\beta}^{\Phi_r}=(-g^{00})(\tfrac{1}{2}\ddot{\Phi}-\tfrac{1}{2}\dot{\Phi}^2-\tfrac{1}{4}g^{00}\dot{g}_{00}\dot{\Phi})+\tfrac{1}{128}\mathrm{Tr}[\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta]-\tfrac{1}{256}(\mathrm{Tr}[\dot{M}_r\eta\dot{M}_r\eta])^2.$$

The $O(d,d)$ invariance is now manifest; as at one loop, when M_r and Φ transform according to Eq. (2.13), $\overline{\beta}^{M_r}$ transforms according to Eq. (2.19), and $\overline{\beta}^{g_r}_{00}$ and $\overline{\beta}^{\Phi_r}$ are invariant.

The calculation of Ref. [8] uses the action corresponding to a scheme with $\lambda=\mu=0$; this action was singled out in Ref. [17] as being the unique ghost-free two-loop string effective action in the presence of torsion, and in the present context appears to lead to the most economical demonstration of invariance, although its use is not mandatory. In this scheme, although there is no need to redefine g_{00}, it is certainly still necessary to include a general g_{00} at intermediate stages of the calculation for the $\overline{\beta}$ functions [due to the $\partial/\partial g_{00}$ term in Eqs. (3.9)], whereas the invariance of the action can be shown with $g_{00}=-1$ throughout. This is a confirmation that the invariance of the $\overline{\beta}$ functions is not simply a consequence of the invariance of the action. Note that the parts of the redefinitions in Eq. (3.11) involving λ and μ are essentially undoing those in Eq. (3.3). Nevertheless, it is a valuable exercise to perform the calculation for general λ and μ because it gives a foretaste of the kinds of field redefinition we shall be obliged to use in the three-loop case.

IV. THE THREE-LOOP CASE

In this section, we shall show the $O(d,d)$ invariance of the $\overline{\beta}$ functions for fields only depending on t at three loops in the absence of torsion. At three loops, the β functions for a general theory are given by [18]

$$\beta^{g(3)}_{\mu\nu}=\tfrac{1}{8}\nabla_\rho R_{\mu\sigma\kappa\lambda}\nabla^\rho R_\nu{}^{\sigma\kappa\lambda}-\tfrac{1}{16}\nabla_\mu R_{\rho\sigma\kappa\lambda}\nabla_\nu R^{\rho\sigma\kappa\lambda}-\tfrac{1}{2}R_{\mu\rho\sigma\tau}R_{\nu\kappa\lambda}{}^\tau R^{\rho\lambda\sigma\kappa}-\tfrac{3}{8}R_{\mu\rho\sigma\nu}R^{\rho\kappa\lambda\tau}R^\sigma{}_{\kappa\lambda\tau}\tag{4.1}$$

and [19]

$$\beta^{\Phi(3)}=-\tfrac{3}{16}R^{\mu\lambda\rho\sigma}R^\nu{}_{\lambda\rho\sigma}\nabla_\mu\nabla_\nu\phi+\tfrac{1}{32}R_{\mu\nu\rho\sigma}R^{\rho\sigma\kappa\lambda}R_{\kappa\lambda}{}^{\mu\nu}-\tfrac{1}{24}R_{\mu\nu\rho\sigma}R^{\kappa\sigma\lambda\nu}R_\kappa{}^\rho{}_\lambda{}^\mu,\tag{4.2}$$

with [19]

$$S^{(3)}_\mu=\tfrac{1}{64}\nabla_\mu(R_{\kappa\lambda\rho\sigma}R^{\kappa\lambda\rho\sigma}).\tag{4.3}$$

Specializing to $g_{\mu\nu}$ as in Eq. (2.7), we find, with the help of various identities given in the Appendix,

$$\overline{\beta}^{G(3)} = \tfrac{1}{32}\{-Y^2+X^3-2WX^2W-XW^2X+2YXW+2YWX-2YW^3-WXWX+2WXW^3+\{\mathrm{tr}[XW^2]-\tfrac{3}{4}\,\mathrm{tr}[X^2]-\tfrac{3}{8}\,\mathrm{tr}[W^4]$$
$$-\tfrac{1}{8}\,(\mathrm{tr}[W^2])^2\}W^2+\tfrac{3}{2}\,\mathrm{tr}[X^2-XW^2+\tfrac{1}{4}\,W^4]X+\mathrm{tr}[W^2](XW^2-\tfrac{1}{4}\,W^4-\tfrac{3}{4}\,X^2)+\tfrac{3}{4}\,(\mathrm{tr}[XW-W^3])(W^3-2XW)$$
$$+\tfrac{1}{8}\,(-\mathrm{tr}[XW]\mathrm{tr}[W^2]+\mathrm{tr}[W^5]-4\mathrm{tr}[XY]+2\mathrm{tr}[YW^2]+\tfrac{5}{2}\,\mathrm{tr}[W^2]\mathrm{tr}[W^3]+14\mathrm{tr}[X^2W]-13\mathrm{tr}[XW^3])W\}+\text{transpose}, \tag{4.4}$$

$$\overline{\beta}^{G(3)}_{00} = \tfrac{1}{64}\,\mathrm{tr}[-18X^3+4XZ-2ZW^2+4Y^2+13YW^3-14Y(XW+WX)+46X^2W^2+13XWXW-39XW^4+6W^6]$$
$$+\tfrac{1}{256}\,\mathrm{tr}[W^2]\mathrm{tr}[4YW-22XW^2+4X^2+13W^4]+\tfrac{1}{64}\,\mathrm{tr}[XW-W^3]\mathrm{tr}[3XW-2W^3]. \tag{4.5}$$

$$\overline{\beta}^{\Phi(3)} = \tfrac{1}{64}\,\mathrm{tr}[2Y^2+2X^3-4Y(XW+WX)+4YW^3+3XWXW-2XW^4-\tfrac{2}{3}\,W^6]$$
$$+\tfrac{1}{128}\,\mathrm{tr}[2YW^2-4XY+14X^2W-13XW^3+W^5]\dot{\Phi}+\tfrac{3}{128}\,\mathrm{tr}[4X^2-4XW^2+W^4]\ddot{\Phi}-\tfrac{1}{256}\,\mathrm{tr}[4XW^2-3W^4]\mathrm{tr}[W^2]$$
$$+\tfrac{1}{128}\,(3\mathrm{tr}[W^3]^2-7\mathrm{tr}[XW]\mathrm{tr}[W^3]+4\mathrm{tr}[XW]^2)-\tfrac{1}{256}\,\mathrm{tr}[2XW-5W^3]\mathrm{tr}[W^2]\dot{\Phi}, \tag{4.6}$$

where $Y=G^{-1}(d^3/dt^3)G$, $Z=G^{-1}(d^4/dt^4)G$. Once again, the $O(d,d)$ invariance is far from manifest, and we are obliged to resort to field redefinitions. Using redefinitions as in Eqs. (A11)–(A14), we find variations of the $\overline{\beta}$ functions as in Eqs. (A15)–(A18). Taking the particular values of the coefficients as given in Eq. (A19), we find, on combining Eqs. (A15)–(A18) with Equations (4.4)–(4.6), and substituting in Eq. (2.15), that we can write

$$\overline{\beta}^{M_r(3)} = \tfrac{1}{32}\,(M_r^{(3)}\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta M_r+\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta M_r^{(3)}\eta M_r-\ddot{M}_r\eta\ddot{M}_r\eta\ddot{M}_r)-\tfrac{1}{16}\,M_r^{(3)}\eta\ddot{M}_r\eta\dot{M}_r$$
$$-\tfrac{1}{32}\,(\ddot{M}_r\eta\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta M_r+\dot{M}_r\eta\dot{M}_r\eta\ddot{M}_r\eta\ddot{M}_r\eta M_r)-\tfrac{3}{64}\,(\ddot{M}_r\eta\dot{M}_r\eta\ddot{M}_r\eta\dot{M}_r\eta M_r+\dot{M}_r\eta\ddot{M}_r\eta\dot{M}_r\eta\ddot{M}_r\eta M_r)$$
$$+\tfrac{1}{32}\,\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\ddot{M}_r\eta M_r-\tfrac{3}{32}\,\dot{M}_r\eta\ddot{M}_r\eta\ddot{M}_r\eta\dot{M}_r\eta M_r+\tfrac{3}{16}\,(\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r$$
$$+\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta M_r)+\tfrac{1}{128}\,\mathrm{tr}[\dot{M}_r\eta\dot{M}_r\eta](\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r+\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta M_r)$$
$$+\tfrac{1}{32}\,(\mathrm{tr}[M_r^{(3)}\eta\dot{M}_r\eta+\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta]+\tfrac{1}{4}\,\mathrm{tr}[\dot{M}_r\eta\dot{M}_r\eta]^2)[\ddot{M}_r+\dot{M}_r\eta\dot{M}_r\eta M_r]$$
$$-\tfrac{1}{16}\,\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta](M_r^{(3)}+3\ddot{M}_r\eta\dot{M}_r\eta M_r+\tfrac{33}{16}\,\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r)+\tfrac{1}{256}\,(\mathrm{tr}[16M_r^{(3)}\eta\ddot{M}_r\eta-61\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta]$$
$$-4\mathrm{tr}[\dot{M}_r\eta\dot{M}_r\eta]\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta])\dot{M}_r+\text{transpose}, \tag{4.7}$$

$$\beta^{g(3)}_{00} = -\tfrac{33}{128}\,\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta\ddot{M}_r\eta\dot{M}_r\eta]+\tfrac{59}{128}\,\mathrm{tr}[M_r^{(3)}\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta]+\tfrac{3}{32}\,\mathrm{tr}[M_r^{(3)}\eta M_r^{(3)}\eta]+\tfrac{15}{128}\,\mathrm{tr}[(\dot{M}_r\eta)^6]$$
$$-\tfrac{1}{256}\,\mathrm{tr}[\dot{M}_r\eta\dot{M}_r\eta]\mathrm{tr}[\ddot{M}_r\eta\ddot{M}_r\eta]-\tfrac{9}{256}\,(\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta])^2, \tag{4.8}$$

$$\beta^{\Phi(3)} = -\tfrac{29}{256}\,\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta\dot{M}_r\eta]\dot{\Phi}-\tfrac{1}{512}\,\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta]\mathrm{tr}[\dot{M}_r\eta\dot{M}_r\eta]\dot{\Phi}+\tfrac{1}{512}\,\mathrm{tr}[(\dot{M}_r\eta)^4]\mathrm{tr}[(\dot{M}_r\eta)^2]-\tfrac{1}{384}\,\mathrm{tr}[(\dot{M}_r\eta)^6]$$
$$+\tfrac{3}{256}\,(\mathrm{tr}[\ddot{M}_r\eta\dot{M}_r\eta])^2-\tfrac{1}{512}\,(\mathrm{tr}[\dot{M}_r\eta\dot{M}_r\eta])^3, \tag{4.9}$$

where $M^{(3)}=(d^3/dt^{(3)})M$. In these equations M_r is defined by Eq. (2.11), but with $B=0$. Once again, the $O(d,d)$ invariance is now manifest; when M and Φ transform according to Eq. (2.13), $\overline{\beta}^{M_r}$ transforms according to Eq. (2.19), and $\overline{\beta}^{(g_r)}_{00}$ and $\overline{\beta}^{\Phi_r}$ are invariant. We have tried to choose the field redefinitions in order to minimize the number of terms which appear here—clearly with only partial success. However, we should stress that it is very nontrivial and apparently miraculous that an $O(d,d)$-invariant form could be found at all, since there are many more constraints than there are free parameters.

V. CONCLUSIONS

In this paper we have shown explicitly that the conformal invariance conditions are form invariant under $O(d,d)$ transformations up to two loops for the general case with torsion. In principle, it should be possible to compare our results with the two-loop results of Ref. [13] which were established for a specific background. We have also demonstrated the $O(d,d)$ invariance up to three loops in the torsion-free case. We should mention, however, that we expect at three loops that in the presence of torsion, assuming that $O(d,d)$ invariance is still preserved, the various $\overline{\beta}$ functions will still adopt the form of Eqs. (4.7)–(4.9), but with M_r now including B as in Eq. (2.11). The three-loop β function in the presence of torsion has been calculated [20], but evidently the inclusion of torsion in the present computation would be prohibitively complex. Finally, it is interesting that we found it essential to keep g_{00} as a variable during the calculation, even though we set $g_{00}=-1$ at the end of the calculation. It is not clear how this is accounted for in the argument for all orders $O(d,d)$ invariance presented in Ref. [7], where the gauge $g_{00}=-1$ was chosen from the outset. Presumably one could in fact demonstrate $O(d,d)$ invariance with a general t-independent g_{00}, without setting $g_{00}=-1$ at the end, though this would be somewhat tedious.

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APPENDIX

In this appendix we list various identities which were useful in our calculations. First, here are the results we used to express the two-loop β functions in terms of W, $\widetilde{W}$, etc., in the two loop case:

$$
\begin{aligned}
R_{i\mu\nu\rho}R_j{}^{\mu\nu\rho}&=\tfrac{1}{4}\,[2X^2-XW^2-W^2X]_{ij}+\tfrac{1}{8}\,\mathrm{tr}[W^2]W^2_{ij},\\
\nabla_iH_{\alpha\beta\gamma}\nabla_jH^{\alpha\beta\gamma}&=-\tfrac{3}{4}\,\mathrm{tr}[\widetilde{W}^2]W^2_{ij}+\tfrac{3}{2}\,(W\widetilde{W}^2W)_{ij},\\
\nabla_\mu H_{i\nu\rho}\nabla^\mu H_j{}^{\nu\rho}&=-\tfrac{1}{4}\,\mathrm{tr}[\widetilde{W}^2]W^2_{ij}-\tfrac{1}{2}\,\mathrm{tr}[W^2]\widetilde{W}^2_{ij}+\tfrac{1}{2}\,(W^2\widetilde{W}^2+\widetilde{W}^2W^2-W\widetilde{W}W\widetilde{W}-\widetilde{W}W\widetilde{W}W-W\widetilde{W}^2W+2\widetilde{X}W\widetilde{W}\\
&\quad+2\widetilde{X}\widetilde{W}W+2W\widetilde{W}\widetilde{X}+2\widetilde{W}W\widetilde{X}-4\widetilde{X}^2)_{ij},\\
R^\mu{}_{ij\nu}H_{\mu\alpha\beta}H^{\nu\alpha\beta}&=\tfrac{1}{4}\,\mathrm{tr}[\widetilde{W}^2](-2X_{ij}+W^2_{ij})+\tfrac{1}{2}\,(W\widetilde{W}^2W)_{ij}-\tfrac{1}{2}\,\mathrm{tr}[\widetilde{W}^2W]W_{ij},\\
R^\mu{}_{\nu\rho i}H_{j\sigma\mu}H^{\nu\rho\sigma}&=\tfrac{1}{4}\,(-2X\widetilde{W}^2+W^2\widetilde{W}^2-W\widetilde{W}W\widetilde{W})_{ij},\\
R_{\mu\nu\rho\sigma}H_i{}^{\mu\rho}H_j{}^{\nu\sigma}&=(\widetilde{W}X\widetilde{W}-\tfrac{1}{2}\,\widetilde{W}W^2\widetilde{W})_{ij},\\
H_{\mu\nu\rho}H_\sigma{}^{\nu\rho}H^\mu{}_{i\tau}H_j{}^{\tau\sigma}&=2\widetilde{W}^4_{ij}+\mathrm{tr}[\widetilde{W}^2]\widetilde{W}^2_{ij},\\
H_{i\mu\nu}H_{j\rho\sigma}H_\tau{}^{\mu\rho}H^{\tau\nu\sigma}&=2\widetilde{W}^4_{ij},
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
R_{\alpha\beta\gamma i}\nabla^\beta H_j{}^{\alpha\gamma}&=\tfrac{1}{8}\,(2W^2\widetilde{X}+2X\widetilde{W}W+2XW\widetilde{W}-4X\widetilde{X}-W^2\widetilde{W}W+W\widetilde{W}W^2-\mathrm{tr}[W^2]W\widetilde{W})_{ij},\\
\nabla_\alpha(H_{i\beta\gamma}H_j{}^{\beta\delta})H_\delta{}^{\alpha\gamma}&=(2W\widetilde{W}^3-\widetilde{W}^2W\widetilde{W}+\widetilde{W}W\widetilde{W}^2-2\widetilde{X}\widetilde{W}^2)_{ij},\\
\nabla_\alpha(H_\beta{}^{\gamma\delta}H_{\gamma\delta i})H_j{}^{\beta\alpha}&=(2\widetilde{X}\widetilde{W}^2+2\widetilde{W}\widetilde{X}\widetilde{W}-W\widetilde{W}^3-2\widetilde{W}W\widetilde{W}^2-\tfrac{1}{2}\,\mathrm{tr}[\widetilde{W}^2]W\widetilde{W})_{ij},\\
\nabla_\alpha H_{ij\beta}H^\alpha{}_{\rho\sigma}H^{\beta\rho\sigma}&=\tfrac{1}{2}\,\mathrm{tr}[\widetilde{W}^2](W\widetilde{W}+\widetilde{W}W-2\widetilde{X})_{ij}+(W\widetilde{W}^3+\widetilde{W}^3W-\mathrm{tr}[W\widetilde{W}^2]\widetilde{W})_{ij},
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
R_{0\mu\nu\rho}R_0{}^{\mu\nu\rho}&=-\tfrac{1}{8}\,\mathrm{tr}[4X^2-4XW^2+W^4],\\
\nabla_0H_{\mu\nu\rho}\nabla_0H^{\mu\nu\rho}&=3\nabla_\mu H_{0\nu\rho}\nabla^\mu H_0{}^{\nu\rho}=3\,\mathrm{tr}[\widetilde{X}^2-\widetilde{X}(W\widetilde{W}+\widetilde{W}W)+\tfrac{1}{2}\,(W\widetilde{W}W\widetilde{W}+W^2\widetilde{W}^2)],\\
R^\mu{}_{00\nu}H_{\mu\rho\sigma}H^{\nu\rho\sigma}&=2R^\mu{}_{\nu\rho0}H_{0\sigma\mu}H^{\nu\rho\sigma}=\mathrm{tr}[X\widetilde{W}^2-\tfrac{1}{2}\,W^2\widetilde{W}^2],\\
R_{\mu\nu\rho\sigma}H_0{}^{\mu\rho}H_0{}^{\nu\sigma}&=-\tfrac{1}{4}\,\mathrm{tr}[\widetilde{W}W\widetilde{W}W],\\
H_{\mu\nu\rho}H_\sigma{}^{\nu\rho}H^\mu{}_{0\tau}H_0{}^{\tau\sigma}&=2H_{0\mu\nu}H_{0\rho\sigma}H_\tau{}^{\mu\rho}H^{\tau\nu\sigma}=-2\,\mathrm{tr}[\widetilde{W}^4].
\end{aligned}
\tag{A3}
$$

Next, here are the identities we used to write $\overline{\beta}^M$ in terms of M at two loops:

$$\dot{M}=\begin{pmatrix} -WG^{-1} & WP-\widetilde{W} \\ G(\widetilde{W}-PW)G^{-1} & G(W+PWP-\widetilde{W}P-P\widetilde{W}) \end{pmatrix},$$

$$\ddot{M}=\begin{pmatrix} AG^{-1} & -AP+B \\ G(PA+B^T)G^{-1} & G(X-B^TP+PB+PAP) \end{pmatrix}, \tag{A4}$$

where

$$P=G^{-1}B, \quad A=2W^2-X, \tag{A5}$$

$$B=2W\widetilde{W}-\widetilde{X},$$

$$\mathrm{Tr}[\dot{M}\,\eta\dot{M}\,\eta]=\mathrm{tr}[W^2-\widetilde{W}^2], \tag{A6}$$

$$\mathrm{Tr}[\dot{M}\,\eta\dot{M}\,\eta\dot{M}\,\eta\dot{M}\,\eta] =\mathrm{Tr}[W^4+\widetilde{W}^4-4W^2\widetilde{W}^2 +2W\widetilde{W}W\widetilde{W}].$$

Now here are the identities for writing the three-loop $\overline{\beta}$ functions in terms of W, etc.,

$$\nabla_\rho R_{i\sigma\kappa\lambda}\nabla^\rho R_j{}^{\sigma\kappa\lambda}=-\tfrac{1}{2}Y^2+\tfrac{1}{2}(YXW+XWY+YWX+WXY)-\tfrac{1}{2}(YW^3+W^3Y)-\tfrac{1}{4}XW^2X-\tfrac{3}{8}WX^2W+\tfrac{1}{4}(W^2X^2 +X^2W^2)-\tfrac{1}{2}(XWXW+WXWX)+\tfrac{5}{16}(WXW^3+W^3XW)-\tfrac{3}{8}W^2XW^2+\tfrac{3}{8}W^6-\tfrac{7}{32}(\mathrm{tr}[W^4]W^2 +\mathrm{tr}[W^2]W^4)-\tfrac{3}{8}\mathrm{tr}[W^2]X^2-\tfrac{1}{4}\mathrm{tr}[X^2]W^2+\tfrac{3}{8}\mathrm{tr}[XW^2]W^2+\tfrac{1}{4}\mathrm{tr}[W^2](XW^2+W^2X)-\tfrac{7}{16}\mathrm{tr}[W^3]W^3 +\tfrac{1}{4}\mathrm{tr}[W^3](XW+WX)+\tfrac{3}{8}\mathrm{tr}[XW]W^3-\tfrac{1}{8}\mathrm{tr}[XW](XW+WX),$$

$$\nabla_i R_{\rho\sigma\kappa\lambda}\nabla_j R^{\rho\sigma\kappa\lambda}=\tfrac{1}{2}WX^2W-\tfrac{1}{4}(W^3XW+WXW^3)-\tfrac{1}{2}W^2XW^2+\tfrac{1}{2}W^6-\tfrac{1}{8}(\mathrm{tr}[W^2]W^4+\mathrm{tr}[W^4]W^2)-\tfrac{1}{2}\mathrm{tr}[X^2]W^2 +\tfrac{1}{2}\mathrm{tr}[XW^2]W^2-\tfrac{1}{4}\mathrm{tr}[W^3]W^3+\tfrac{1}{2}\mathrm{tr}[XW]W^3,$$

$$R_{i\rho\sigma\tau}R_{j\kappa\lambda}{}^\tau R^{\rho\lambda\sigma\kappa}=-\tfrac{1}{8}X^3+\tfrac{1}{16}XW^2X+\tfrac{1}{16}(X^2W^2+W^2X^2)-\tfrac{1}{16}(WXWX+XWXW)-\tfrac{1}{32}W^2XW^2+\tfrac{1}{32}(WXW^3 +W^3XW)+\tfrac{1}{32}W^6-\tfrac{1}{32}\mathrm{tr}[W^2]W^4-\tfrac{1}{64}\mathrm{tr}[W^4]W^2+\tfrac{1}{64}(\mathrm{tr}[W^2])^2W^2+\tfrac{1}{16}\mathrm{tr}[XW](WX+XW) +\tfrac{1}{64}\mathrm{tr}[W^3]W^3-\tfrac{1}{16}\mathrm{tr}[XW]W^3-\tfrac{1}{32}\mathrm{tr}[W^3](XW+WX),$$

$$R_{i\kappa\lambda j}R^{\kappa\rho\sigma\tau}R^\lambda{}_{\rho\sigma\tau}=\tfrac{1}{4}(\mathrm{tr}[X^2]-\mathrm{tr}[XW^2]+\tfrac{1}{4}\mathrm{tr}[W^4])(\tfrac{1}{2}W^2-X)+\tfrac{1}{32}\mathrm{tr}[W^2]W^4+\tfrac{1}{8}(\mathrm{tr}[XW^3]-\mathrm{tr}[X^2W] -\tfrac{1}{4}\mathrm{tr}[W^2]\mathrm{tr}[W^3])W+\tfrac{1}{8}WX^2W-\tfrac{1}{16}(W^3XW+WXW^3),$$

$$\nabla_i\nabla_j(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})=-\tfrac{1}{4}(4\mathrm{tr}[XY]-8\mathrm{tr}[X^2W]-2\mathrm{tr}[YW^2]+7\mathrm{tr}[XW^3] -\mathrm{tr}[W^5]+\mathrm{tr}[XW]\mathrm{tr}[W^2]-\mathrm{tr}[W^3]\mathrm{tr}[W^2]), \tag{A7}$$

$$\nabla_\rho R_{0\sigma\kappa\lambda}\nabla^\rho R_0{}^{\sigma\kappa\lambda}=\tfrac{1}{2}\mathrm{tr}[Y^2]+\mathrm{tr}[XWXW]+\tfrac{3}{8}\mathrm{tr}[W^6]-\mathrm{tr}[YXW]-\mathrm{tr}[YWX]+\mathrm{tr}[YW^3]+\tfrac{7}{8}\mathrm{tr}[X^2W^2]-\tfrac{7}{4}\mathrm{tr}[XW^4] +\tfrac{1}{16}\mathrm{tr}[W^4]\mathrm{tr}[W^2]+\tfrac{1}{8}\mathrm{tr}[W^2]\mathrm{tr}[X^2]+\tfrac{1}{16}(\mathrm{tr}[W^3])^2-\tfrac{1}{8}\mathrm{tr}[W^3]\mathrm{tr}[XW]-\tfrac{1}{8}\mathrm{tr}[W^2]\mathrm{tr}[XW^2],$$

$$\nabla_0 R_{\rho\sigma\kappa\lambda}\nabla_0 R^{\rho\sigma\kappa\lambda}=\mathrm{tr}[Y^2]-2\mathrm{tr}[YXW]-2\mathrm{tr}[YWX]+2\mathrm{tr}[YW^3]+2\mathrm{tr}[XWXW]+\tfrac{3}{2}\mathrm{tr}[X^2W^2]-3\mathrm{tr}[XW^4]+\tfrac{1}{2}\mathrm{tr}[W^6] +\tfrac{1}{4}\mathrm{tr}[X^2]\mathrm{tr}[W^2]-\tfrac{1}{2}\mathrm{tr}[XW^2]\mathrm{tr}[W^2]+\tfrac{1}{4}\mathrm{tr}[W^4]\mathrm{tr}[W^2]+\tfrac{1}{4}(\mathrm{tr}[W^3-XW])^2, \tag{A8}$$

$$R_{0\rho\sigma\tau}R_{0\kappa\lambda}{}^\tau R^{\rho\lambda\sigma\kappa}=\tfrac{1}{32}\mathrm{tr}[4X^3-6X^2W^2+2XWXW+XW^4]-\tfrac{1}{64}\{(\mathrm{tr}[W^3])^2-4\mathrm{tr}[XW]\mathrm{tr}[W^3]+4(\mathrm{tr}[XW])^2\},$$

$$R_{0\kappa\lambda 0}R^{\kappa\rho\sigma\tau}R^\lambda{}_{\rho\sigma\tau}=\tfrac{1}{4}\mathrm{tr}[X^3]-\tfrac{3}{8}\mathrm{tr}[X^2W^2]+\tfrac{1}{8}\mathrm{tr}[XW^4]+\tfrac{1}{16}\mathrm{tr}[W^2]\mathrm{tr}[XW^2]-\tfrac{1}{32}\mathrm{tr}[W^2]\mathrm{tr}[W^4],$$

$$\nabla_0\nabla_0(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})=\tfrac{1}{2}\{4\mathrm{tr}[XZ]+4\mathrm{tr}[Y^2]-14\mathrm{tr}[YWX]-14\mathrm{tr}[YXW]-8\mathrm{tr}[X^3]+30\mathrm{tr}[X^2W^2]+15\mathrm{tr}[XWXW] -2\mathrm{tr}[ZW^2]+13\mathrm{tr}[YW^3]-33\mathrm{tr}[XW^4]+5\mathrm{tr}[W^6]+\mathrm{tr}[YW]\mathrm{tr}[W^2]+\mathrm{tr}[X^2]\mathrm{tr}[W^2]+2(\mathrm{tr}[XW])^2 -5\mathrm{tr}[XW^2]\mathrm{tr}[W^2]-4\mathrm{tr}[XW]\mathrm{tr}[W^3]+3\mathrm{tr}[W^4]\mathrm{tr}[W^2]+2(\mathrm{tr}[W^3])^2\}.$$

The following are the identities needed to write $\overline{\beta}^M$ in terms of M at three loops [here M is defined by Eq. (2.11), but with $B=0$]:

$$M^{(3)}\eta\ddot{M}\eta\dot{M}=\begin{pmatrix}(Y-3WX-3XW+6W^3)XW & 0\\ 0 & Y(-X+2W^2)W\end{pmatrix},$$

$$M^{(3)}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta M=\begin{pmatrix}(Y-3WX-3XW+6W^3)W^3 & 0\\ 0 & YW^3\end{pmatrix},$$

$$\ddot{M}\eta\ddot{M}\eta\ddot{M}=\begin{pmatrix}X^3-2X^2W^2-2W^2X^2+4W^2XW^2 & 0\\ 0 & -X^3+2XW^2X\end{pmatrix},$$

$$\ddot{M}\eta\ddot{M}\eta\dot{M}\eta\dot{M}\eta M=\begin{pmatrix}X^2W^2 & 0\\ 0 & X^2W^2-2XW^4\end{pmatrix},$$

$$\ddot{M}\eta\dot{M}\eta\ddot{M}\eta\dot{M}\eta M=\begin{pmatrix}XWXW-2W^3XW-2XW^4+4W^6 & 0\\ 0 & XWXW\end{pmatrix},$$

$$\ddot{M}\eta\dot{M}\eta\dot{M}\eta\ddot{M}\eta M=\begin{pmatrix}XW^2X-2W^4X & 0\\ 0 & XW^2X-2XW^4\end{pmatrix},$$

$$\dot{M}\eta\ddot{M}\eta\ddot{M}\eta\dot{M}\eta M=\begin{pmatrix}WX^2W-2WXW^3 & 0\\ 0 & WX^2W-2W^3XW\end{pmatrix},$$

$$\ddot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}=\begin{pmatrix}-XW^4+2W^6 & 0\\ 0 & XW^4\end{pmatrix},$$

$$\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta M=\begin{pmatrix}-W^6 & 0\\ 0 & -W^6\end{pmatrix}, \tag{A9}$$

$$\mathrm{tr}[M^{(3)}\eta M^{(3)}\eta]=2\,\mathrm{tr}[-Y^2+3WXY+3XWY-6W^3Y],$$

$$\mathrm{tr}[M^{(3)}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta]=2\,\mathrm{tr}[YW^3-3XW^4+3W^6],$$

$$\mathrm{tr}[\ddot{M}\eta\ddot{M}\eta\dot{M}\eta\dot{M}\eta]=2\,\mathrm{tr}[X^2W^2-2XW^4]$$

$$\mathrm{tr}[\ddot{M}\eta\dot{M}\eta\ddot{M}\eta\dot{M}\eta]=2\,\mathrm{tr}[XWXW-2XW^4+2W^6],$$

$$\mathrm{tr}[\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta\dot{M}\eta]=-2\,\mathrm{tr}[W^6],$$

$$\mathrm{tr}[M^{(3)}\eta\dot{M}\eta]=2\,\mathrm{tr}[-YW+3XW^2-3W^4]. \tag{A10}$$

Finally, the following is the general form of field redefinition we need to consider in Eq. (3.8) up to three loops:

$$\delta G=\tfrac{1}{4}(-g^{00})W^2+\alpha_1X^2+\alpha_2(XW^2+W^2X)+\alpha_3WXW+\alpha_4W^4+[\beta_1\mathrm{tr}(W^3)+\beta_2\mathrm{tr}(XW)]W+\mathrm{tr}(W^2)(\gamma_1W^2+\gamma_2X)$$
$$+\dot{g}^{00}(\lambda_1(XW+WX)+\lambda_2W^3), \tag{A11}$$

$$\delta g_{00}=(-g^{00})[\epsilon_1\mathrm{tr}(W^4)+\epsilon_2\mathrm{tr}(XW^2)+\epsilon_3\mathrm{tr}(X^2)]+\dot{g}_{00}[\tfrac{1}{2}\epsilon_2\mathrm{tr}(W^3)+\epsilon_3\mathrm{tr}(XW)], \tag{A12}$$

$$\delta\Phi=\kappa_1\mathrm{tr}[W^4]+\kappa_2\mathrm{tr}[XW^2]+\kappa_3\mathrm{tr}[X^2]+\kappa_4(\mathrm{tr}[W^2])^2+\dot{g}_{00}(\mu_1\mathrm{tr}[XW]+\mu_2\mathrm{tr}[W^3]), \tag{A13}$$

$$(\delta S)_0=\delta_1\mathrm{tr}[W^5]+\delta_2\mathrm{tr}[XW^3]+\delta_3\mathrm{tr}[X^2W]+\delta_4\mathrm{tr}[XY]+\delta_5\mathrm{tr}[YW^2]+\delta_6\mathrm{tr}[W^3]\mathrm{tr}[W^2]+\delta_7\mathrm{tr}[XW]\mathrm{tr}[W^2]$$
$$+\tfrac{1}{4}[\epsilon_1\mathrm{tr}(W^4)+\epsilon_2\mathrm{tr}(XW^2)+\epsilon_3\mathrm{tr}(X^2)]\mathrm{tr}(W). \tag{A14}$$

These redefinitions lead using Eq. (3.9) to changes in the $\overline{\beta}$ functions given by

$$\delta\overline{\beta}^{G(3)}=\{-\tfrac{1}{2}\alpha_1Y^2+(\tfrac{1}{2}\alpha_1-\alpha_2+\tfrac{1}{16})YXW+(\tfrac{1}{2}\alpha_1-\alpha_2)YWX+(\tfrac{1}{16}-\alpha_3)WYX+(2\alpha_2-\tfrac{1}{16})YW^3+(\alpha_3-\tfrac{1}{16})WYW^2$$
$$-(\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_3)X^3+(\alpha_1+\tfrac{3}{2}\alpha_3-\alpha_4-\tfrac{1}{8})X^2W^2+(\alpha_2+\tfrac{1}{2}\alpha_3-\alpha_4-\tfrac{5}{32})XWXW+\tfrac{1}{4}(3\alpha_1+6\alpha_2-2\alpha_4-\tfrac{1}{16})XW^2X$$
$$-\tfrac{1}{4}(\alpha_1-2\alpha_2+2\alpha_3+2\alpha_4+\tfrac{3}{16})WX^2W-(\alpha_1+\alpha_2-3\alpha_4-\tfrac{1}{4})XW^4-\tfrac{1}{2}(2\alpha_2-\alpha_3-4\alpha_4-\tfrac{3}{8})WXW^3$$
$$+(\alpha_2-\alpha_3+\alpha_4+\tfrac{1}{8})W^2XW^2-\tfrac{1}{2}(2\alpha_2+\alpha_3+6\alpha_4+\tfrac{7}{16})W^6-\tfrac{1}{4}\mathrm{tr}[W^2](\alpha_1X^2+(2\alpha_2-\tfrac{1}{16})XW^2+(\alpha_3-\tfrac{1}{16})WXW$$
$$+(\alpha_4+\tfrac{3}{32})W^4\}+\text{transpose}$$
$$+[3\beta_1\mathrm{tr}(-X^2W+2XW^3-W^5)+\beta_2\mathrm{tr}(-XY+YW^2-W^5+XW^3)]W+[(\delta_1'+\epsilon_1)\mathrm{tr}(W^5)$$
$$+(\delta_2'-\epsilon_1+\tfrac{3}{4}\epsilon_2)\mathrm{tr}(XW^3)+(\delta_3'-\tfrac{1}{2}\epsilon_2+\tfrac{1}{2}\epsilon_3)\mathrm{tr}(X^2W)+(\delta_4'-\tfrac{1}{2}\epsilon_3)\mathrm{tr}(XY)+(\delta_5'-\tfrac{1}{4}\epsilon_2)\mathrm{tr}(YW^2)+\delta_6'\mathrm{tr}(W^3)\mathrm{tr}(W^2)$$
$$+\delta_7'\mathrm{tr}(XW)\mathrm{tr}(W^2)]W+[3\beta_1\mathrm{tr}(XW^2-W^4)+\beta_2\mathrm{tr}(YW+X^2-2XW^2)](W^2-X)+\tfrac{1}{2}[\epsilon_1\mathrm{tr}(W^4)+\epsilon_2\mathrm{tr}[XW^2]$$
$$+\epsilon_3\mathrm{tr}(X^2)](W^2-X)-\tfrac{1}{2}[\beta_1\mathrm{tr}(W^3)+\beta_2\mathrm{tr}(XW)]\mathrm{tr}(W^2)W+\mathrm{tr}(-X^2+2XW^2-W^4)(\gamma_1W^2+\gamma_2X)$$
$$-\tfrac{1}{64}\mathrm{tr}[4X^2-4XW^2+W^4]W^2, \tag{A15}$$

where

$$\delta_1'=\delta_1-2\kappa_1,\quad \delta_2'=\delta_2+2\kappa_1-\tfrac{3}{2}\kappa_2,$$
$$\delta_3'=\delta_3+\kappa_2-\kappa_3,\quad \delta_4'=\delta_4+\kappa_3,$$
$$\delta_5'=\delta_5+\tfrac{1}{2}\kappa_2,\quad \delta_6'=\delta_6-2\kappa_4,\quad \delta_7'=\delta_7+2\kappa_4, \tag{A16}$$

$$\delta\overline{\beta}^{g(3)}_{00}=\tfrac{1}{2}\mathrm{tr}\{(-\alpha_1+2\alpha_2+2\alpha_3-8\delta_2'+8\delta_3'-8\epsilon_2+10\epsilon_3-\tfrac{35}{16})X^2W^2+(-\alpha_1+2\alpha_2-4\delta_2'+4\delta_3'-\epsilon_2+2\epsilon_3-\tfrac{11}{16})XWXW$$
$$+(2\alpha_2+\alpha_3-4\delta_2'+12\delta_5'+4\epsilon_1-2\epsilon_2-\tfrac{1}{2})YW^3+(\alpha_1-4\delta_3'+4\delta_4'-4\delta_5'+\epsilon_2-2\epsilon_3+\tfrac{1}{4})(YXW+XYW)$$
$$+(-6\alpha_2-3\alpha_3+4\alpha_4-20\delta_1'+16\delta_2'-12\epsilon_1+10\epsilon_2-4\epsilon_3+\tfrac{31}{8})XW^4+(-4\alpha_4+20\delta_1'+8\epsilon_1-2\epsilon_2-\tfrac{3}{2})W^6$$
$$+(-4\delta_5'+\epsilon_2)ZW^2+(-4\delta_4'+2\epsilon_3)ZX+(-4\delta_3'-6\epsilon_3+\tfrac{1}{2})X^3-4\delta_4'Y^2+(\beta_1+2\gamma_1-8\delta_6'-\tfrac{1}{2}\epsilon_2)\mathrm{tr}[W^3][XW-W^3]$$
$$+(3\beta_1+2\gamma_1-12\delta_6)\mathrm{tr}[XW^2-W^4]\mathrm{tr}[W^2]+(\beta_2+2\gamma_2-8\delta_7'-\epsilon_3)\mathrm{tr}[XW][XW-W^3]$$
$$+(\beta_2-4\delta_7')\mathrm{tr}[YW+X^2-2XW^2]\mathrm{tr}[W^2]+\gamma_2\mathrm{tr}[YW-XW^2]\mathrm{tr}[W^2]-\tfrac{1}{2}\mathrm{tr}(W^2)[\epsilon_1\mathrm{tr}(W^4)+\epsilon_2\mathrm{tr}[XW^2]+\epsilon_3\mathrm{tr}(X^2)]\}, \tag{A17}$$

$$\delta\overline{\beta}^{\Phi(3)}=\mathrm{tr}[-(\kappa_2+2\kappa_3)X^3-\kappa_3Y^2+(\kappa_3-\kappa_2)(YXW+YWX)+2\kappa_2YW^3+(\kappa_2-2\kappa_1)XWXW-(4\kappa_1-2\kappa_2-3\kappa_3)X^2W^2$$
$$-(\kappa_2+2\kappa_3-12\kappa_1+\tfrac{1}{32})XW^4-(6\kappa_1+\kappa_2-\tfrac{1}{32})W^6]-\tfrac{1}{2}\mathrm{tr}[W^2]\mathrm{tr}([\kappa_1+\tfrac{1}{16}]W^4+[\kappa_2-\tfrac{1}{16}]XW^2+\kappa_3X^2)$$
$$-2\kappa_4(\mathrm{tr}[X^2-2XW^2+W^4]\mathrm{tr}[W^2]+2(\mathrm{tr}[XW-W^3])^2+(\mathrm{tr}[W^2])^3)+\dot{\Phi}[(\delta_1'+\epsilon_1)\mathrm{tr}(W^5)+(\delta_2'-\epsilon_1+\tfrac{3}{4}\epsilon_2)\mathrm{tr}(XW^3)$$
$$+(\delta_3'-\tfrac{1}{2}\epsilon_2+\tfrac{1}{2}\epsilon_3)\mathrm{tr}(X^2W)+(\delta_4'-\tfrac{1}{2}\epsilon_3)\mathrm{tr}(XY)+(\delta_5'-\tfrac{1}{4}\epsilon_2)\mathrm{tr}(YW^2)+\delta_6'\mathrm{tr}(W^3)\mathrm{tr}(W^2)+\delta_7'\mathrm{tr}(XW)\mathrm{tr}(W^2)]$$
$$+\tfrac{1}{4}\mathrm{tr}[Y-3XW+2W^3]\mathrm{tr}[2(\kappa_3-\mu_1)XW+(\kappa_2-2\mu_2)W^3]-\tfrac{1}{2}\mathrm{tr}[XW-W^3]\mathrm{tr}[\mu_1XW+\mu_2W^3]$$
$$-\tfrac{1}{2}[\epsilon_1\mathrm{tr}(W^4)+\epsilon_2\mathrm{tr}(XW^2)+\epsilon_3\mathrm{tr}(X^2)]\ddot{\Phi}, \tag{A18}$$

In order to demonstrate $O(d,d)$ invariance, we find that we need to take

$$\begin{aligned}
&\alpha_1=-\tfrac{1}{16}, \quad \alpha_2=\tfrac{1}{32}, \quad \alpha_3=\tfrac{1}{16}, \quad \alpha_4=-\tfrac{1}{32},\\
&\beta_1=-\tfrac{1}{8}, \quad \beta_2=\tfrac{1}{8}, \quad \gamma_1=\tfrac{7}{64}, \quad \gamma_2=-\tfrac{1}{8},\\
&\lambda_1=-\tfrac{1}{2}\alpha_1, \quad \lambda_2=-\left(\alpha_2+\tfrac{1}{2}\alpha_3\right),\\
&\delta_1=\tfrac{3}{16}, \quad \delta_2=0, \quad \delta_3=-\tfrac{15}{64}, \quad \delta_4=\tfrac{3}{32}, \quad \delta_5=-\tfrac{3}{64},\\
&\delta_6=-\tfrac{7}{256}, \quad \delta_7=\tfrac{1}{64},\\
&\epsilon_1=\tfrac{3}{64}, \quad \epsilon_3=-\epsilon_2=\tfrac{3}{16},\\
&\kappa_1=\tfrac{1}{128}, \quad \kappa_3=-\kappa_2=\tfrac{1}{32}, \quad \kappa_4=-\tfrac{1}{128}, \quad \mu_1=\kappa_3, \quad \mu_2=\tfrac{1}{2}\kappa_2.
\end{aligned} \tag{A19}$$

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